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# Capital for Non-Performing Loans

**Rafael Weißbach**[1*], **Carsten von Lieres und Wilkau**[2]

[1] Institut für Wirtschafts- und Sozialstatistik, Technische Universität Dortmund, Dortmund, Germany e-mail: Rafael.Weissbach@uni-dortmund.de
[2] Credit Risk Management, WestLB AG, Düsseldorf, Germany

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**Abstract** A portfolio of non-performing loans needs economic capital. We present two models for forecasting the non-performing portfolio's loss and derive the probability distribution. In the first model, the loss for each loan is a Gaussian random variable, and the risk determinants are the portfolio concentration, as well as systematic and idiosyncratic risk. Our second model allows for diversification with a performing portfolio, because an investor typically owns a combination of performing and non-performing loans. This model is a mixture model. For both models, formulae for the economic capital and the fair contribution of a single loan are given. We calibrate the models with times series data and a benchmark portfolio. Our main finding is that the credit portfolio risk of non-performing loans depends on the volatility of economic activity, on the granularity of the portfolio and on the performing portfolio. Finally, we compare the economic capital charges for non-performing loans from our models with the regulatory capital charges of Basel II. The main difference is that our capital charges are sensitive to economic activity volatility, whereas the regulatory ones are not.

**Key words** Portfolio credit risk, non-performing loan, recovery, Basel II

* JEL subject classifications. C51, G11, G18, G33
*Correspondence to*: Rafael Weißbach, Institut für Wirtschafts- und Sozialstatistik, Technische Universität Dortmund, 44221 Dortmund, Germany
Phone: +49 231 755 5419, Fax: +49 231 755 5284

## 1 Introduction

Regulators have recently made an effort to align capital requirements with actual credit risk. Nevertheless, regulatory capital does not acknowledge portfolio diversification and hence, internal credit portfolio models for the calculation of economic capital for a specific portfolio remain necessary. Regulatory capital is defined for each single counterpart in the portfolio and is calculated as a percentage of its exposure at default (EAD). Economic capital, as calculated with a portfolio model, is a measurement which applies to the entire portfolio. For the single counterpart, the allocation procedures decompose the portfolio capital into risk contributions, the capital charges, but, in contrast to regulatory capital charges, usually not proportionally to the exposure.

We are interested in a particular type of counterpart, the non-performing counterpart. Finding economic capital for non-performing exposure is our primary aim, the secondary aim is to compare it with the regulatory capital. We assume that the portfolio contains only loans. This seems to be very restrictive, however, many financial products are comparable to loans when it comes to portfolio credit risk, even derivative products.

The risk associated with the non-performing loan lies in the uncertainty relating to the loss caused by the default, in common glossary the loss given default (LGD). It is that part of the exposure which cannot be regained in the course of settling the claims.[1] This risk is rarely short-term, because a bank is usually exposed to a counterpart years after the default definition is fulfilled, especially for private debt portfolios where the trading of non-performing debt is rare (Carey (1998)). We will show that for credit portfolio models with a risk horizon of one year, portfolio credit risk is significant.

Non-performing loans have been met with considerable interest in the literature. Schuermann (2005), for example, found that financial intermediates are typically exposed two to four years after the last cash is paid. Gupton et al. (2000) found an expectation of one and a half years for large bank loans. As a consequence, one clear observation in the banking sector is that the losses vary materially from what is expected (Gupton et al. (2000)).[2] However, our literature survey found only two studies on economic capital for non-performing loans. Tasche (2004) derives the capital analytically as a by-product, based on a zero-inflation model for loss deriving from the individual performing loan. Gupton et al. (1997) apply simulation techniques and assumes the loss given default to operate as a random multiplier of the expected exposure, once the counterpart has defaulted. To our surprise, we did not find any study on non-performing loans using the popular mixture model methodology (see e.g. McNeil et al. (2005); Credit Suisse First Boston (CSFB) (1997)).

The paper present two key ideas. On one hand, we wish to calculate the economic capital by modelling the loss given default with a Gaussian model, allowing

[1] We define exposure as the unsecured portion of the outstanding amount, net of risk mitigation, but gross of collateralization.

[2] See also Calem and LaCour-Little (2004) for a justification of randomness of loss, given foreclosure for mortgage loans. More studies are cited in Gupton et al. (1997).

for correlation between the loan losses within the non-performing portfolio. On the other hand, we extend the mixture model proposed by Bürgisser et al. (2001) and find an analytical method of reflecting the risk-mitigating effect of diversification with a portfolio of performing loans.

Our contribution is to provide (i) economic capital formulae for non-performing loans, (ii) calibration of the parameters these formulae depend on, and (iii) a numerical comparison of the economic with the regulatory capital requirement percentages in an impact study.

Specifically, we find that formulae for economic capital that take diversification with performing loans into account, differ substantially from those without diversification. Mathematically, diversification is considerably more difficult and requires a mixture model, whereas without diversification, a Gaussian model is sufficient.

For the calibration, our general finding is that we can calibrate two types of parameters separately, some do not depend on the specific portfolio composition, call them portfolio-independent, and some do, call them portfolio-specific. With diversification, the portfolio-specific parameters are separable into those for the non-performing portfolio and those for the performing portfolio.

The model comparison shows that taking diversification into account changes the capital charges for non-performing loans. Interestingly, the differences depend on the loss volatility and the magnitude of expected losses. Not surprisingly, regulatory capital ignores the sensitivity of capital to the volatility of loan losses, although different loss expectations are considered in the approach based on internal ratings.

The paper is organized as follows. Our Gaussian model for the non-performing portfolio's loss is developed in section 2. The model in section 3 allows for diversification with the performing portfolio. We calibrate the two models in section 4. Section 5 compares the quantitative impact of our two economic models and the two regulatory models of Basel II. Section 6 discusses and concludes.

## 2 Gaussian model

Assume you own a non-performing portfolio which we refer to as portfolio $\mathcal{N}$. You are interested in the loss forecast for a one-year horizon from today. Each loan $A$ in the portfolio had a default event at some elapsed point in time. At default, the exposure of the loan was found to be $e_A$. Usually, the portfolio owner has, until now, defined provisions or write-offs directly at default and both positive and negative provisions in the course of settling the claims. The future loss for a certain loan $A$ will be the difference between the exposure in one year's time and today's exposure, it will be the change in provision $\Delta_A$. Now, we will be defining a model for the change, relative to the exposure at default $e_A$:

$$\delta_A = \frac{\Delta_A}{e_A} \tag{1}$$

There are two reasons for modelling the loss incurred through a loan relative to its exposure at default and not relative to the current exposure: first of all, to avoid a dominance of small loans, potentially leading to unreasonably large changes relative to the current exposure, during the calibration (see Section 4), and secondly, to enable a comparison with the Basel II regulations and with our second model.

The portfolio loss is

$$L_{\mathcal{N}} = \sum_{A \in \mathcal{N}} \Delta_A = \sum_{A \in \mathcal{N}} e_A \delta_A.$$

It is a well-established assumption in the modelling of portfolio risk that losses are not stochastically independent. In our case of non-performing loans, the stochastic component is the loss given default and positive correlations between those losses have been reported by Gupton et al. (2000). We model the dependence with

$$\delta_A = Y + \varepsilon_A \quad A \in \mathcal{N}. \tag{2}$$

The influence of the economy is modeled by $Y$ and the ideosyncratic risk of a loan $A$ is reflected in $\varepsilon_A$ We assume $Y$ and the $\varepsilon_A$'s to be independent. The dependence between two losses is determined by the variance of $Y$ and the variances of $\varepsilon_A$'s. We assume all $\varepsilon_A$ to have the same variance $\sigma_\varepsilon^2$. This assumption implies the variance of the provision change $\sigma_\delta^2 = Var(\delta_A)$ and the the correlation $\rho = corr(\delta_A, \delta_B) = \sigma_Y^2/(\sigma_Y^2 + \sigma_\varepsilon^2)$ to be equal for any loan $A$. Furthermore, we assume the loss expectation to be 0, otherwise further write-offs would have been made. For simplicity, we assume Gaussian distributions for $Y$ and the $\varepsilon_A$'s.

Clearly, the portfolio's loss expectation is 0 and, as $L_{\mathcal{N}}$ is the sum of Gaussian random variables, it is itself Gaussian. The distribution is specified in full by calculating the loss variance

$$Var(L_{\mathcal{N}}) = \left( \sum_{A \in \mathcal{N}} e_A^2 + \sum_{A,B \in \mathcal{N}, A \neq B} \rho e_A e_B \right) \sigma_\delta^2 \approx e^2 (H + \rho) \sigma_\delta^2. \tag{3}$$

Here $e = \sum_{A \in \mathcal{N}} e_A$ denotes the total exposure of the non-performing portfolio and $H = (\sum_{A \in \mathcal{N}} e_A^2)/e^2$ denotes the Herfindahl-Hirschmann index of portfolio concentration (Hirschmann (1964)).[3] In the limiting case of an infinitesimally granular portfolio, $H$ is 0 and the variance reduces to the systematic effect of $Y$, namely $e^2 \sigma_Y^2$ and is positive if $\rho > 0$.

The economic capital at level $\gamma$ is now given by

$$EC_{\mathcal{N},\gamma} = e u_\gamma (H + \rho)^{1/2} \sigma_\delta, \tag{4}$$

where $u_\gamma$ denotes the $\gamma$-quantile of the standard normal distribution. Typical values for $\gamma$ are $99.95, 99.9, 99.5, 99.0$ and $90\%$ their $u_\gamma$'s are $3.29, 3.09, 2.58, 2.33$, and $1.28$.

Apart from the calibration of parameters (see Section 4), Formula (4) constitutes a stand-alone economic capital calculation for a portfolio of non-performing loans. This is one of the aims of the paper. The model incorporates a single-name concentration penalty, reflected by the Herfindahl-Hirschmann index $H$. However,

[3] Its is easy to see that $\sum_{A,B \in \mathcal{N}, A \neq B} e_A e_B / e^2 \approx 1$.

we demonstrate in our subsequent impact study, that the systematic effect of $Y$ dominates the overall risk, an effect usually found in financial portfolio risk modelling (Gordy (2000)). Only if the correlation $\rho$ is negligible, does the portfolio composition play a crucial role.

An objective that we have not accomplished so far is the following. Once a portfolio owner knows the amount of capital needed to prevent a portfolio from default, he will be interested in assigning the responsibility for that capital to the single loans, i.e. in defining capital charges, especially in order to calculate risk-adjusted performance measures. If all loans in the portfolio are similar, each will carry a similar portion. But what if this is not the case? In other words, the economic capital needs to be allocated. An obvious approach, given the attribution logic for regulatory capital in Basel II, is to define the risk weight as the exposure at default and define the capital charge[4]

$$ec_A = \frac{e_A EC_{\mathcal{N},\gamma}}{e} = e_A u_\gamma (H + \rho)^{1/2} \sigma_\delta. \tag{5}$$

As a concluding remark, we wish to emphasis that both, the model and the resulting formulae for portfolio capital and capital charge of the single loan, are very accessible. The loss mechanic is simple and, as a result, the calibration - which may already be foreclosed - is simple. Hence, frequent up-dates, which are essential in a rapidly changing economic environment, are possible.

## 3 Mixture model

In this section we consider the sub-additivity question. That is, for consistent risk measures, the sum of the economic capital for two portfolios is larger than the total economic capital for the two portfolios. Often, an owner of a non-performing portfolio will also own a performing portfolio. In order to save capital, he needs to know the extent to which the economic capital for a non-performing portfolio stand-alone is overstated. This may be accomplished by calculating the economic capital for the joint portfolio.

Such an extension of the model comes at a price. In order to handle the increased complexity associated with incorporating the performing portfolio, we must reduce the accuracy of our model for the non-performing portfolio. Instead of the relative change in provision over the risk horizon $\delta_A$, as given in (1), we now ignore the risk horizon and use the loss given default over the entire (individual) settlement period, including the initial provision, and denote it as $\lambda_A$. The defect occurring through this simplification will be corrected subsequently in the course of calibrating the model.

[4] A conceptually more advanced idea is to use the derivative of the loss variance with respect to the single exposure (Credit Suisse First Boston (CSFB) (1997)). However, it can be seen that model (2) implies that the derivative of the loss variance is linear in terms of exposure (for an infinitesimally granular portfolio). Hence, an exposure-linear attribution is appealing.

An additional simplification is that, instead of the individual loss given default modelled in (2), we ignore single-name concentration in the non-performing portfolio and restrict ourselves to the systematic effect. We do so by omitting the individual loss given default, namely the $\varepsilon_A$ in model (2). This is feasible, because, as mentioned earlier, diversification causes the impact of the individual noise on the economic capital to be negligible. As the systematic loss given default $Y$ in model (2) is different from the loss given default here, because of the first simplification, we now denote the systematic effect as $\Lambda$ and use a one-factor model for the loss given default proposed by Bürgisser et al. (2001):

$$\lambda_A = l_A \Lambda \tag{6}$$

Here, $\Lambda$ is a random variable with expectation 1 and variance $\sigma_\Lambda^2$, so that $l_A$ represents the expected loss given default for loan $A$.

The loss from the non-performing portfolio $\mathcal{N}$ is now expressed as

$$L_{\mathcal{N}} = \sum_{A \in \mathcal{N}} e_A \lambda_A.$$

where $e_A$ is again the exposure at default. We now add the loss of the performing portfolio to the loss of the non-performing portfolio. The default is a Bernoulli event, denoted here as $I_A$, and the loss from the performing portfolio can be written as

$$L_{\mathcal{P}} = \sum_{A \in \mathcal{P}} e_A \lambda_A I_A.$$

Due to the nature of our model (6) for $\lambda_A$, this can be represented as $\Lambda L_{\mathcal{P}}^d$ where $L_{\mathcal{P}}^d = \sum_{A \in \mathcal{P}} e_A l_A I_A$ is the portfolio loss of a performing portfolio with the loss given default known in advance. It is important to mention that the defaults in this model should not be assumed as independent. The dependence formulation we use is that $I_A$ follows a Bernoulli event with a default probability $\mu_A$.[5] Such probabilities all depend on a latent random variable $X$, e.g.:

$$\mu_A = p_A(X) = \begin{cases} p_A X & \text{CreditRisk}^+ \\ \Phi\left(\frac{\Phi^{-1}(p_A) + \sqrt{r}X}{\sqrt{1-r}}\right) & \text{CreditMetrics} \end{cases} \tag{7}$$

Here, $r$ is the asset correlation and conditional of $X$ the defaults are independent. We mix the distribution of the portfolio loss - given the loss given default - with the distribution of $\Lambda$, this is known as mixture modelling (see McNeil et al., 2005, pg. 352).[6]

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[5] Note that any loss model that is linear in the loss given default can be used, including e.g. contagion models (Davis and Lo (2001)). For an evidence of contagion in financial markets see e.g. Caporale et al. (2005).

[6] Including the effect of rating migrations is possible in a simulation, e.g. if they follow a homogeneous Markov process. For a comment on the homogeneity see Kiefer and Larson (2007); Weißbach and Dette (2007).

Clearly, the loss for the non-performing and the performing portfolio together is given by

$$L_{\mathcal{N}\cup\mathcal{P}} = L_{\mathcal{N}} + L_{\mathcal{P}}.$$

The calculation of the distribution of $L_{\mathcal{N}\cup\mathcal{P}}$ and therefore those for economic capital require only elementary probability theory. The only additional requirement we assume in the calculations is the conditional independence of the randomness in the loss given default modelled by $\Lambda$, to the probability of default driver $X$, as well as to the defaults $I_A$ themselves. With respect to the applicability of the independence assumption of default and loss given default, there is a controversy in the literature. In a large study, Altman et al. (2002) proved small positive correlations for default rates and loss given default rates. So did Frye (2000) as well as Hu and Perraudin (2002). However, the latter refer to Carey and Gordy (2001) who found negligible correlations. Some indication of negative correlation may be found in Carey (1998). Despite the stronger evidence of positive correlation, we assume independence. This enables a closed-form solution of the economic capital and hence a fast computational evaluation. Details are in the Appendix (see A.1) and the result is:

$$\begin{aligned} EC_{\mathcal{N}\cup\mathcal{P},\gamma} &= CreditVaR_{\mathcal{N}\cup\mathcal{P},\gamma} - E(L_{\mathcal{P}}) - E(L_{\mathcal{N}}) \\ &= CreditVaR_{\mathcal{N}\cup\mathcal{P},\gamma} - E(\Lambda L^d_{\mathcal{P}}) - E(L_{\mathcal{N}}) \end{aligned}$$

Because of the independence assumption, we can write

$$E(\Lambda L^d_{\mathcal{P}}) = E(\Lambda)E(L^d_{\mathcal{P}}) = E(L^d_{\mathcal{P}})$$

so that

$$EC_{\mathcal{N}\cup\mathcal{P},\gamma} = CreditVaR_{\mathcal{N}\cup\mathcal{P},\gamma} - \left(\sum_{A\in\mathcal{P}} p_A e_A l_A + \sum_{A\in\mathcal{N}} e_A l_A\right).$$

This may be written explicitly as

$$\begin{aligned} EC_{\mathcal{N}\cup\mathcal{P},\gamma} = \inf &\left\{k : \sum_{n\geq 1} P\left(L^d_{\mathcal{P}} = n - \sum_{A\in\mathcal{N}} e_A l_A\right) P\left(\Lambda \leq \frac{k}{n}\right) > \gamma\right\} \\ &- \left(\sum_{A\in\mathcal{P}} p_A e_A l_A + \sum_{A\in\mathcal{N}} e_A l_A\right) \end{aligned} \tag{8}$$

Although this is a lengthy expression, it is easy to evaluate. The probability distribution function for $L^d_{\mathcal{P}}$ is assumed to be known (see (7)). The distribution of $\Lambda$, however, must be selected.[7]

As in the stand-alone model of section 2, we need to attribute the capital to the responsible loans and define capital charges. A standard procedure is to consider the portfolio loss variance $Var(L_{\mathcal{N}\cup\mathcal{P}})$ as a risk measure and attribute the risk

[7] In the following section we argue that there is a generalized Beta-distribution.

caused by the change in variance as the net exposure $\nu_A = e_A l_A$ changes (Credit Suisse First Boston (CSFB) (1997)). The formula for $Var(L_{\mathcal{N}\cup\mathcal{P}})$ is given in the Appendix (see A.2). The risk contribution is the marginal effect of the presence of $e_A$ on the standard deviation of the loss distribution and can be written as $RC_A = \frac{e_A}{2\sigma}\frac{\partial Var}{\partial e_A}$, where the sum of the risk contribution is $\sum_A RC_A = \sigma$. So we can write instead

$$ec_A = \frac{e_A EC_{\mathcal{N}\cup\mathcal{P},\gamma}}{2Var(L_{\mathcal{N}\cup\mathcal{P}})}\frac{\partial Var(L_{\mathcal{N}\cup\mathcal{P}})}{\partial e_A}$$

which is an additive risk attribution. A short calculation yields the economic capital charge for a non-performing loan

$$ec_A = \frac{EC_{\mathcal{N}\cup\mathcal{P},\gamma}}{Var(L_{\mathcal{N}\cup\mathcal{P}})} e_A l_A \left(E(L^d_{\mathcal{P}}) + E(L_{\mathcal{N}})\right)\sigma^2_{\Lambda}. \tag{9}$$

This representation includes a penalty for a large single exposure at default $e_A$, reflected by the quadratic component. Note that $e_A$ is also contained in $E(L_{\mathcal{N}})$. As pointed out at the beginning of this section, diversification between the performing and the non-performing portfolio is possible, and the interaction between the portfolio becomes manifest in the joint parameters. More specifically, the expected loss of the performing portfolio $E(L^d_{\mathcal{P}})$ influences the economic capital charge of a non-performing loan.

In the next section, we calibrate the models derived in the current and preceding sections. A detailed interpretation of the differences between the economic capital charges in the two models will follow.

## 4 Calibration of the models

The capital charges of our economic models are influenced by parameters that depend on the specific portfolio and those that are portfolio-independent. We first estimate the independent parameters for both models and then discuss the portfolio-specific parameters. Table 1 lists all relevant parameters and gives their estimates.

### *4.1 Portfolio-independent parameters*

We start with the Gaussian model, which contains two parameters in the Formula (5) for the economic capital of non-performing loans that do not depend on the actual portfolio. In the first place, again, the unexpected changes in the loss given default, or, to be exact for this model, the changes in provision, influence the capital via the parameter $\sigma^2_\delta$. Secondly, the correlation between changes in provision, namely $\rho$, is also independent of the portfolio.

We estimate $\sigma^2_\delta$ by studying provision changes $\Delta_A$ over several years. The change, relative to the exposure at default, i.e. $\delta_A$, is assumed to follow Model (2), hence we observe

$$\delta_{ti} = Y_t + \varepsilon_{ti}, \quad t = 1, \ldots, T, \;\; i = 1, \ldots, n_t.$$

**Table 1** Parameters in the Gaussian and the mixture model and their estimates

| Portfolio-independent Parameters | | | | | | |
|---|---|---|---|---|---|---|
| Gaussian Model | | Mixture Model | | | | |
| | | | $\Lambda \sim Beta_{[a,b]}(\alpha,\beta)$ | | | |
| $\sigma_\delta$ | $\rho$ | $\sigma^2_\Lambda$ | $a$ | $b$ | $\alpha$ | $\beta$ |
| 12% | 15% | 10% | 0.05 | 2.4 | 5.3 | 7.8 |
| **Portfolio-specific Parameters** | | | | | | |
| Gaussian Model | | Mixture Model | | | | |
| $H$ | | $D_\gamma$ | | | | |
| 0.25% - 2.5%† | | 11.6-13.7‡ | | | | |

† Depends on the amount of diversification in the non-performing portfolio.
‡ Depends on the diversification of non-performing and performing portfolio as well as on whether the expected loss from the non-performing portfolio is approximated.

This model, with unobservable $Y_t$'s, is known as the one-way random effects model and the maximum likelihood estimation is thus established (see Arnold, 1981, pg. 245ff). Based on a microeconomic study of 120 losses observed over the seven years from 1998 to 2004, we estimate $\sigma_\delta$ as given in Table 1.[8] The risk-increasing effect of estimation uncertainty (Rosenow and Weißbach (2005)) is ignored here.

The second portfolio-independent parameter in the Gaussian model is $\rho$. Again using results for the one-way random effects model, the estimator for $\rho$ is given in Arnold (1981) and our estimate is given in Table 1.

In the mixture model, the systematic loss given default $\Lambda$ depends mainly on the macroeconomy and not on the actual portfolio composition. We need to specify the distribution of $\Lambda$. However, we distinguish between the variance $\sigma^2_\Lambda$, which enters directly into the capital charge Formula (9) for a non-performing loan, and the shape, which enters into the distribution of the portfolio loss, as is demonstrated in due course. For the first stage we consider the variance.

We estimate $\sigma^2_\Lambda$ by studying ratios between the (final) loss given default $\lambda_A$ and (ex ante) expected loss given default $l_A$ for the same losses as in the Gaussian model. Clearly, this measurement includes idiosyncratic effects, but we can easily identify the variance of the systematic effect in the model

$$\lambda_{ti} = \Lambda_t + \varepsilon_{ti}, \quad t = 1, \ldots, T, \;\; i = 1, \ldots, n_t.$$

The maximum likelihood estimation for the unobservable $\Lambda$ is given in Arnold (1981). Based on the former microeconomic study of 120 losses, we estimate $\sigma^2_\Lambda$ as given in Table 1.[9]

8 In order to reduce the effect of small loans, we use weights proportional to the exposure. Additionally, we adapt the balanced design of the simple randem effects model to our unbalanced data.

9 Again, we adapt the estimator for weights proportional to the exposure and the unbalanced design.

At the second stage, we specify the distribution of $\Lambda$. It influences the economic capital for the entire portfolio with the Formula (9), namely $EC_{\mathcal{N}\cup\mathcal{P},\gamma}$. There are several proposals in the literature. (Bürgisser et al. (2001)) uses a log-normal distribution, which unfortunately implies the possibility of infinite loss rates for a given exposure at default. Empirically, even a bimodal distribution is possible (Schuermann (2005)). The Beta distribution is a generalization of the uniform distribution, used by Tasche (2004) for the loss given default and also used in commercial models for the recovery rate, e.g. in CreditMetrics (Gupton et al. (1997)). We prefer the latter distribution because, based on the data from WestLB, we found, using exploratory methods (see Weißbach (2006)), that the generalized Beta distribution fits the distribution of $\Lambda$. Specifically, as the distribution for the factor $\Lambda$, we assume a generalization of the Beta distribution, i.e.

$$\Lambda \sim a + (b - a)\ Beta(\alpha, \beta),$$

where $0 \leq a < 1 < b$ and $\alpha, \beta > 0$. Accounting for the estimate of $\sigma^2_\Lambda$, we find, using the least-squares method, the estimates listed in Table 1.

### *4.2 Portfolio-specific parameters*

If the portfolio is not assumed to be infinitely granular, it is clear that the portfolio composition influences the amount of capital needed for the portfolio. In the Gaussian model, only the composition of the non-performing portfolio has an impact on the capital (see Formula again (5)), whereas both the performing and the non-performing portfolios influence the mixture model capital (see Formula again (9)).

We start again with the Gaussian model. The capital charge depends on the Herfindahl-Hirschmann index $H$, which measures diversification in the non-performing portfolio. We designed several non-performing portfolios and Table 1 shows the resulting indices. We will see in the following section that this parameter is almost negligible and, therefore, we refrain from reporting the detailed portfolios any further.

In the mixture model, we now need to specify the factor

$$EC_{\mathcal{N}\cup\mathcal{P},\gamma}(E(L^d_{\mathcal{P}}) + E(L_{\mathcal{N}}))/Var(L_{\mathcal{N}\cup\mathcal{P}}).$$

For simplicity, we denote it by $D_\gamma$, it represents the influence of the portfolio on the capital charge for the single loan. The factor may be simulated using Monte Carlo techniques. However, there are simplifications we used in order to reduce the computational complexity. First of all, one may assume that the non-performing loan was once a performing loan. The migration from performing to non-performing, i.e. the default, is described mainly by the probability of default. One can show that the expected loss of the non-performing portfolio equals the expected loss of the performing portfolio (see Appendix A.3).

In order to calibrate the portfolio factor $D_\gamma$, we design a benchmark portfolio and a diversified version of that portfolio. Each consists of a performing portfolio

**Table 2** Composition of benchmark portfolio $\mathcal{P}_{benchmark}$ for determining portfolio factor $D_\gamma$.

| Exposure | Number | Fraction | Probability of Default |
|---|---|---|---|
| Huge (200-1,000 mio$^*$) | 150 | 3% | 0.03% |
| Large (30-60 mio$^*$) | 350 | 7% | 0.03%-0.07%$^*$ |
| Mediocre (0.3-30 mio$^*$) | 4,000 | 80% | 0.07%-2%$^*$ |
| Small (0.1-0.3 mio$^*$) | 500 | 10% | 2% - 7%$^*$ |

$^*$ Randomly drawn from uniform distribution with limits as specified.

$\mathcal{P}_{benchmark}$ and $\mathcal{P}_{diversified}$ as well as a non-performing element. For the non-performing portfolios, we only need to know their expected loss (see Formula (8)). The result that the expected loss of the non-performing portfolio is approximately the expected loss of the performing portfolio makes the explicit portfolio composition of the non-performing portfolio obsolete. The performing portfolio composition for the benchmark portfolio with five thousand loans is found in Table 2. The diversified portfolio $\mathcal{P}_{diversified}$ contains the same number of loans, but with exposure randomly drawn between 1 and 100 million currency units. For modelling default and estimating the defaults probabilities see Weißbach et al. (2008). The expected default probabilities range randomly between 0.03% and 7%.

Using CreditRisk+ (at level 99.9%), that is the first default model in (7), and the portfolio-independent parameters calibrated at the beginning of this section, the portfolio factor is 11.66 for the benchmark and 11.72 for the diversified portfolio. Interestingly, the degree of diversification in the performing portfolio merely changes the portfolio factor.

There is another simplification that may be useful for the practitioner. The ratio of economic capital and loss variance is essentially the same for the performing portfolio, as for the entire portfolio. We verify the assumption with the two calibration portfolios. For the benchmark performing portfolio, we find that $EC_{\mathcal{P},99.9\%}/Var(L_{\mathcal{P}}) = 0.04$. For the entire portfolio the ratio is not very different, namely $EC_{\mathcal{N}\cup\mathcal{P},99.9\%}/Var(L_{\mathcal{N}\cup\mathcal{P}}) = 0.035$. For the diversified portfolio, the ratios are 0.0026 and 0.0022. This justifies the simplification of $D_\gamma$ to $2EC_\gamma(L^d_{\mathcal{P}})E(L^d_{\mathcal{P}})/Var(L^d_{\mathcal{P}})$. The simplified portfolio factor for the benchmark portfolio is 13.48 and for the diversified 13.74; the inaccuracy of around 15% appears to be acceptable. The range of the parameter estimates over all situations is listed in Table 1.

## 5 Comparing economic and regulatory models

The primary aim of this section is to compare numerically the capital charges for a non-performing loan derived in Section 2 with the results from Section 3. The secondary aim is a comparison of our economic view with the regulatory view.

We have identified typical parameters in the course of the calibration in section 4, however, for example, the exposure at default $e_A$ and the expected loss given default $l_A$ clearly vary from loan to loan. Additionally, the level of diversification

**Table 3** Charges for economic and regulatory capital as percentages of the exposure at default.

| Economic Capital | | Regulatory Capital (Basel II) | |
|---|---|---|---|
| Gaussian Model | Mixture Model | Standardized Approach | IRB |
| $\Phi^{-1}(\gamma)\sqrt{H+\rho}\,\sigma_\delta$ | $(D_\gamma\sigma_A^2 - 1)l_A$ | 8% | $l_{A,\gamma} - l_A$ |

of a portfolio can vary and the volatility of the loss given default is also difficult to generalize as it depends on the estimation method, the data used and the portfolio aimed at. As a consequence, we will consider a broader range of numerical situations in order to discuss the models.

The foremost common quantification of capital requirement is the formulation as "percentage of the exposure at default" and we reformulate our results in these percentages and present them in Table 3.

In fact, the capital charge for the Gaussian model (see Formula (5)) is already given as such a percentage, if we ignore the first factor $e_A$.

For the capital charge in the mixture model (given in Formula (9)), we note that - in contrast to the Gaussian model - the loss does include initial provisions. It is assumed here that the initial provision is always equal to the ex-ante expected loss for that specific claim, in conformity with the regulatory requirement on provisions. The capital requirements, net of the initial provision and relative to the exposure at default, is again given in Table 3, where we have made use of the notation $D_\gamma$ as defined in section 4.

Table 3 lists also the regulatory capital charges, namely for the standardized approach and the internal ratings-based approach (IRB).

For the standardized approach, the regulations on "past due loans" (Basel Committee on Banking Supervision (2004), paragraph 75) prescribe risk weights (net of specific provision and partial write-offs) of 100% in most cases. Eight percent of the risk weighted assets add to the regulatory capital.

For the internal ratings-based approach (IRB), the regulations need interpretation. Basel II, paragraph 471, advises estimating the loss given default for any exposure "reflect[ing] the possibility that the bank would have to recognize additional, unexpected losses during the recovery period". We can quantify the impact of acknowledging the unexpected loss given default by applying Formula (10) (see Appendix A.1). For our two calibration portfolios from section 4, we find that the economic capital (at level $\gamma = 99.9\%$ used in Basel II (Basel Committee on Banking Supervision (2004))) is $20\%$ times higher than for the use of the expected loss given default as a *deterministic* forecast. Hence, applying the 20% surcharge directly to a deterministic loss given default is exactly what the regulator requires (see Basel Committee on Banking Supervision (2004), paragraph 272), hence $l_{A,99.9\%} = 1.2\ l_A$.[10] The resulting capital requirement for defaulted exposure is denoted by $l_{A,\gamma} - l_A$ in Table 3.

[10] The loss given default influences the economic capital linearly.

**Table 4** Capital charges - relative to the exposure at default - for non-performing loans, dependent on the expectation and the variance of the loss given default (LGD)$^{\dagger}$: Comparison of Gaussian model$^{\ddagger}$, mixture model, IRB approach and the standardized approach of Basel II.

| Risk factor | | Model | | | |
|---|---|---|---|---|---|
| | | Economic capital charge | | Basel II charge | |
| E(LGD) | Var(LGD) | Gaussian$_{H=2.5\%}^{H=0.25\%}$ | Mixture | IRB | Standardized |
| 20% | 7% | $5.7\%_{+0.2}^{-0.1}$ | 0% | 4% | 8% |
| | 10% | $6.8\%_{+0.3}^{-0.1}$ | 3% | 4% | 8% |
| | 13% | $7.8\%_{+0.3}^{-0.2}$ | 10% | 4% | 8% |
| 45% | 7% | $12.8\%_{+0.6}^{-0.3}$ | 0% | 9% | 8% |
| | 10% | $15.3\%_{+0.7}^{-0.3}$ | 7% | 9% | 8% |
| | 13% | $17.4\%_{+0.8}^{-0.4}$ | 23% | 9% | 8% |
| 70% | 7% | $19.9\%_{+0.9}^{-0.5}$ | 0% | 14% | 8% |
| | 10% | $23.8\%_{+1.1}^{-0.6}$ | 11% | 14% | 8% |
| | 13% | $27.1\%_{+1.3}^{-0.6}$ | 35% | 14% | 8% |

$^{\dagger}$ Expected loss given default E(LGD) is $l_A$. The variance Var(LGD) given in the table is the mixture model's $\sigma_\Lambda^2$. The equivalent variance for the Gaussian model is defined in the text.
$^{\ddagger}$ The main Herfindahl-Hirschmann index in the Gaussian model is $H = 1\%$, the sensitivity is quantified. The correlation is assumed to be $\rho = 15\%$.

Before we select the numerical examples for the capital requirements in these four models, we need to identify the key risk factors. From the formulae of Table 3, it is evident that the primary factors for the capital requirements are the volatility and expectation of the loss given default. The main differences between these economic models is the diversification mechanism: in the Gaussian model, the diversification in the non-performing portfolio is limited by the correlation $\rho$, whereas, in the mixture model, the diversification across portfolios is measured in the portfolio factor $D_\gamma$. In Table 4, we present capital requirements for a variety of risk-factor situations and a confidence level of $99.9\%$. The correlation $\rho$ is chosen to be 15% (see Section 4). As Schuermann (2005) states, the recoveries are distributed from 30% to 80%, we investigate expected (individual) loss given defaults between 20% and 70%. Our own estimate of $10\%$ for the loss given default variance $\sigma_\Lambda^2$ (see again Section 4) is at the center of the range 7-13%. By comparing the definitions for the loss given default in the Gaussian and the mixture model one recognizes that $\sigma_\delta \approx l_A \sigma_\Lambda$. For a given $\sigma_\Lambda^2$ this gives the corresponding $\sigma_\delta$ for the Gaussian model.

Evidently, the regulatory capital which is necessary to cover the non-performing portfolio is less risk sensitive than our formulae for the economic capital. The standardized approach is the least adaptive. The IRB approach adapts to the expected loss given default $l_A$, in the sense that more expected loss given default requires more capital. The mixture model adapts for both expected loss given default and loss given default volatility. The same holds for the Gaussian model, but, the latter is less sensitive to loss given default volatility than the mixture model. Interestingly, the Gaussian model depends on the loss given default *volatility*, but the mixture model on the *variance*. In general, the requirements are less widely spread for the Gaussian model than for the mixture model. On average, the level of requirements for the Gaussian model is higher than that of the mixture model (as well as for the regulatory requirements). The reason is twofold. On the one hand, and mainly, the lack in diversification potential with the performing-portfolio leads to an overstatement of the capital requirements. On the other hand, but of less importance, single-name concentration effects are neglected in the integrated model as a result of the definition of the loss given default. Interestingly, the requirements for most situations are lower than the 25.9% reported in Tasche (2004). An additional finding for the Gaussian, stand-alone, capital charges is the very small sensitivity with respect to the portfolio concentration as measured with the Herfindahl-Hirschmann index.

## 6 Discussion and Conclusion

We will now discuss our theses. Our first finding is that banks must hold capital for non-performing loans. Second, for the quantification of the capital, we found that the most demanding prerequisite is the decision about the volatility the loss of a defaulted loan will have. This is not unexpected, but the large sensitivity of the capital charges with respect to the volatility did surprise us. Third, we propose to distinguish between two risks, that of the non-performing portfolio stand-alone and that of the non-performing portfolio as a sub-portfolio of the entire one. Understanding these two risks helps the model to be used effectively in the risk management processes. Interestingly, the two risks require different methodologies, for the stand-alone case it suffices to apply standard theory to Gaussian random variables. However, an application of mixture distribution theory is required for the more complex case of diversification with the performing portfolio.

There are some additional critical points which should be mentioned. One could argue that we synthetically transform a market risk issue into a credit risk issue and thus complicate its measurement and management. The reason for arguing this is that the volatility of the loss given default is due to changes in the market price of collateral. Secondly, if non-performing loans are traded directly after default, our research question is superfluous and no capital needs to be allocated.

Taking these points into account, a portfolio owner must, before applying our risk measures or the empirical results, ask himself several questions. How is he managing his portfolios? Is it feasible to transfer non-performing loans to a unit that has market risk management in place? Is he trading loans or holding them until settlement? What is the scale of the portfolio? Owners of large and medium size portfolios are more likely to earn sufficient profits from accurate models to enforce the methodological development. What is his view on the operational risk in the widely-used Monte-Carlo risk machines? A medium-size portfolio owner might well use the analytical methods we propose.

On balance, our model-based results about economic capital charges for non-performing loans may well be valuable for the portfolio credit risk management under conditions that we consider realistic, especially for European banks.

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## A Appendix

### *A.1 Loss distribution in the mixture model*

The distribution of $L_\mathcal{P}$ in Section 3 can be calculated as

$$\begin{aligned} P(L_\mathcal{P} \leq k) &= P(\Lambda \leq k/L^d_\mathcal{P}), \quad k/0 = \infty \\ &= \sum_{n \geq 0} P(\Lambda \leq k/L^d_\mathcal{P} \mid L^d_\mathcal{P} = n)\; P(L^d_\mathcal{P} = n) \\ &= P(L^d_\mathcal{P} = 0) + \sum_{n \geq 1} P(\Lambda \leq k/n)\; P(L^d_\mathcal{P} = n), \end{aligned} \tag{10}$$

where $n$ denotes an amount of loss and $k$ is a quantile of $L_\mathcal{P}$. In order to stress that $L^d_\mathcal{N}$ is deterministic, we will denote it by $\eta$, so that $L_\mathcal{N} = \Lambda\eta$, because $\lambda_A = l_A \Lambda$. This amounts to $\eta = \sum_{A \in \mathcal{N}} e_A\, l_A$. Define $L^d_{\mathcal{N}\cup\mathcal{P}} = L^d_\mathcal{P} + \eta$ so that $L_{\mathcal{N}\cup\mathcal{P}} = \Lambda L^d_{\mathcal{N}\cup\mathcal{P}}$. The distribution of $L_{\mathcal{N}\cup\mathcal{P}}$ is now - similar to the above calculation -

$$P(L_{\mathcal{N}\cup\mathcal{P}} \leq k) = P(L^d_{\mathcal{N}\cup\mathcal{P}} = 0) + \sum_{n \geq 1} P(\Lambda \leq k/n)\; P(L^d_{\mathcal{N}\cup\mathcal{P}} = n),$$

where the distribution of $L^d_{\mathcal{N}\cup\mathcal{P}}$ - apart from a shift - now depends only on the distribution of $L^d_\mathcal{P}$, because $P(L^d_{\mathcal{N}\cup\mathcal{P}} = n) = P(L^d_\mathcal{P} = n - \eta)$. For $n = 0$, $P(L^d_{\mathcal{N}\cup\mathcal{P}} = 0)$ can only be positive if $\eta = 0$ which is only the case when $\mathcal{N} = \emptyset$. We will not consider this degenerate case. So we can neglect the first term of the above calculation. The credit value-at-risk at level $\gamma$ is then given by

$$CreditVaR_{\mathcal{N}\cup\mathcal{P},\gamma} = \inf\left\{ k : \sum_{n \geq 1} P(L^d_\mathcal{P} = n - \eta)\; P(\Lambda \leq k/n) > \gamma \right\}.$$

The economic capital is

$$EC_{\mathcal{N}\cup\mathcal{P},\gamma} = CreditVaR_{\mathcal{N}\cup\mathcal{P},\gamma} - \sum_{A\in\mathcal{P}} p_A e_A l_A + \sum_{A\in\mathcal{N}} e_A l_A),$$

because

$$E(L_{\mathcal{N}\cup\mathcal{P}}) = E(\Lambda)(E(L^d_{\mathcal{P}}) + \eta) = \sum_{A\in\mathcal{P}} p_A e_A l_A + \sum_{A\in\mathcal{N}} e_A l_A.$$

### *A.2 Loss variance in the mixture model*

The loss variance in section 3 can be calculated as

$$\begin{aligned} Var(L_{\mathcal{N}\cup\mathcal{P}}) &= E(Var(L_{\mathcal{N}\cup\mathcal{P}} \mid \Lambda)) + Var(E(L_{\mathcal{N}\cup\mathcal{P}} \mid \Lambda)) \\ &= E(\Lambda^2 Var(L^d_{\mathcal{P}} + \eta)) + Var(\Lambda E(L^d_{\mathcal{N}\cup\mathcal{P}})) \\ &= E(\Lambda^2)Var(L^d_{\mathcal{P}}) + E(L^d_{\mathcal{N}\cup\mathcal{P}})^2 Var(\Lambda) \\ &= (1+\sigma^2_\Lambda)Var(L^d_{\mathcal{P}}) + \sigma^2_\Lambda(E(L^d_{\mathcal{P}}) + \eta)^2. \end{aligned}$$

With the first model for the dependent probabilities of default in (7) it is $Var(I_A) = E(Var(I_A|X)) + Var(E(I_A|X))$. Clearly, $E(I_A|X) = p_A X$ and $Var(I_A|X) = p_A X(1-p_A X)$ and

$$\begin{aligned} Var(I_A) &= E(p_A X(1-p_A X)) + Var(p_A X) \\ &= p_A E(X - p_A X^2) + p_A^2 Var(X)^2 \\ &= p_A(1 - p_A E(X^2)) + p_A^2 Var(X)^2 \\ &= p_A(1 - p_A(1 + Var(X)^2)) + p_A^2 Var(X)^2. \end{aligned}$$

From the conditional independence of the $I_A$ give $X$ follows

$$Var(L^d_{\mathcal{P}}) = \sum_{A\in\mathcal{P}} e_A^2 l_A^2\, p_A\left(1 - p_A\left(1 + Var(X)^2\right)\right) + Var(X)^2 E(L^d_{\mathcal{P}})^2,$$

with the expected loss of the performing portfolio $E(L^d_{\mathcal{P}}) = \sum_{A\in\mathcal{P}} p_A e_A l_A$.

### *A.3 Expected non-performing loss in the mixture model*

In section 4, we make use of the relationship between the expected loss for the performing portfolio, $E(L^d_{\mathcal{P}}) = E(L_{\mathcal{P}})$, and the expected loss for the non-performing portfolio, $E(L_{\mathcal{N}})$. Note first that the non-performing portfolio $\mathcal{N}$ is the (conditional) portion of a (formerly) performing portfolio $\mathcal{P}$ of expected size $\sum_{A\in\mathcal{P}} p_A$ in number and $\sum_{A\in\mathcal{P}} \lambda_A e_A p_A$ in loss. Than

$$\begin{aligned} E(L_{\mathcal{N}}) &= E(\sum_{A\in\mathcal{N}} \lambda_A e_A) \approx E(E(\sum_{A\in\mathcal{P}} \lambda_A e_A I_A \mid \Lambda)) \\ &= \sum_{A\in\mathcal{P}} l_A e_A p_A = E(L^d_{\mathcal{P}}). \end{aligned}$$